Section 3.9

Logarithmic Differentiation

(1) Review of Logarithm Properties and Derivatives

(2) Logarithmic Differentiation

Review, Logarithmic Functions

Definition of Logarithms

$$y = \log_b(x) \quad \Leftrightarrow \quad b^y = x$$

Basic Properties

(I) $b^{\log_b(x)} = x$ and $\log_b(b^x) = x$.

(II) $\log_b(xy) = \log_b(x) + \log_b(y)$.

(III) $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$.

(IV) $\log_b(x^y) = y\log_b(x)$.

Derivatives

$$\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)} \qquad \frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Examples, Logarithms

(a) $\log_4(16) = \log_4(4^2) = 2$

(b) $\log_4\left(\frac{1}{16}\right) = \log_4(4^{-2}) = -2$

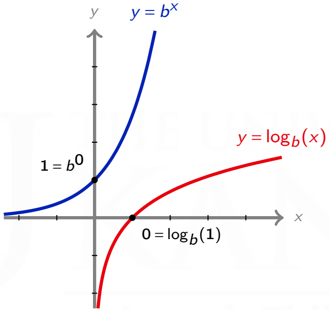
(c) $\log_{1/4}\left(\frac{1}{16}\right) = \log_{1/4}\left((1/4)^2\right) = 2$

(d) $\log_2\left(\sqrt[3]{2}\right) = 1/3$

(e) $\log_2\left(\frac{1}{4\sqrt{2}}\right) = \log_2\left(2^{-5/2}\right) = -5/2$

(f) $\ln\left(\frac{x^2(x-1)}{\sqrt{x+1}}\right) = 2\ln(x) + \ln(x-1) - \frac{1}{2}\ln(x+1)$

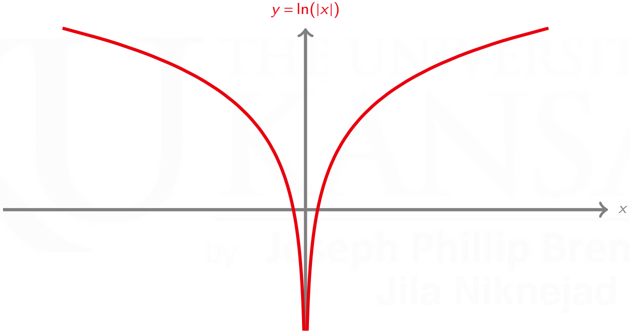
The domain of $\log_b(x)$ is the range of $y = b^x$, namely $(0, \infty)$.



So the identity $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ is valid only if $x > 0$.

Question: How can we extend the logarithm function to something whose derivative is $\frac{1}{x}$ for all nonzero x?

Answer: Define $f(x) = \ln|x| = \begin{cases} \ln(x) & \text{if } x > 0, \\ \ln(-x) & \text{if } x < 0. \end{cases}$



Then $\dfrac{d}{dx}\left(\ln|x|\right) = \begin{cases} \frac{1}{x} & \text{if } x > 0, \\ \frac{-1}{-x} & \text{if } x < 0. \end{cases}$

$$\boxed{\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \text{ for all } x \neq 0.}$$

Logarithmic Differentiation

Example 1: Let $\boxed{y = f(x) = x^x.}$ Find the derivative $f'(x)$.

⚠ The rules $\dfrac{d}{dx}(x^n) = nx^{n-1}$ and $\dfrac{d}{dx}(b^x) = b^x \ln(b)$ do not apply!:

- $f(x) = x^x$ is <u>not a power function</u> (the exponent is not a number)
- $f(x) = x^x$ is <u>not an exponential function</u> (the base is not a number)

Use the technique of logarithmic differentiation.

(I) Take the natural logarithm of both sides of the equation $y = x^x$ and use the Laws of Logarithms to simplify the expression.

(II) Differentiate the expression implicitly with respect to x.

(III) Solve for dy/dx.

Logarithmic Differentiation

Solution:

(I)
$$y = x^x$$
$$\ln(y) = \ln(x^x) = x\ln(x)$$

(II)
$$\frac{y'}{y} = x \cdot \frac{1}{x} + 1 \cdot \ln(x) = 1 + \ln(x)$$

(III)
$$y' = y(1 + \ln(x)) \quad \boxed{= x^x(1 + \ln(x)).}$$

Logarithmic Differentiation

(I) Take the natural logarithm of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify the expression.

(II) Differentiate the expression implicitly with respect to x.

(III) Solve for dy/dx, replacing y with $f(x)$.

Logarithmic Differentiation

Logarithmic Differentiation

(I) Take the natural logarithm of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify the expression.

(II) Differentiate the expression implicitly with respect to x.

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In general there are four cases for exponents and bases:

(1) $\dfrac{d}{dx}\left(a^b\right) = 0$

(2) $\dfrac{d}{dx}\left([f(x)]^b\right) = b[f(x)]^{b-1} f'(x)$

(3) $\dfrac{d}{dx}\left(a^{g(x)}\right) = a^{g(x)} \ln(a) g'(x)$

(4) $\dfrac{d}{dx}\left([f(x)]^{g(x)}\right) = \left(g'(x)\ln(f(x)) + \dfrac{g(x)f'(x)}{f(x)}\right)[f(x)]^{g(x)}$

Example 2, Logarithmic Differentiation

(I) Find the derivative of $y = (2x+1)^5(x^3-1)^3$.

$$\ln(y) = \ln\left((2x+1)^5(x^3-1)^3\right) = 5\ln(2x+1) + 3\ln(x^3-1)$$

$$\text{Differentiating: } \frac{1}{y}\frac{dy}{dx} = 5\frac{2}{2x+1} + 3\frac{3x^2}{x^3-1}$$

$$\frac{dy}{dx} = (2x+1)^5(x^3-1)^3\left(\frac{10}{2x+1} + \frac{9x^2}{x^3-1}\right)$$

(II) Find the derivative of $y = \left(\frac{x^2(x-1)}{\sqrt{x+1}}\right)^\pi$.

$$\ln(y) = \pi\left(2\ln(x) + \ln(x-1) - \frac{1}{2}\ln(x+1)\right)$$

$$\frac{dy}{dx} = \pi\left(\frac{2}{x} + \frac{1}{x-1} - \frac{1}{2(x+1)}\right)\left(\frac{x^2(x-1)}{\sqrt{x+1}}\right)^\pi$$

Logarithmic Differentiation: Example 2:

(III) $y = x^{\sin(x)}$: $\qquad \ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x)\ln(x)$

$$\frac{y'}{y} = \cos(x)\ln(x) + \frac{\sin(x)}{x}$$

$$y' = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$$

(IV) $y = \sin(x)^{\arctan(x)}$: $\qquad \ln(y) = \arctan(x)\ln(\sin(x))$

$$\frac{y'}{y} = \frac{\ln(\sin(x))}{1+x^2} + \frac{\cos(x)}{\sin(x)}\arctan(x)$$

$$y' = \sin(x)^{\arctan(x)}\left(\frac{\ln(\sin(x))}{1+x^2} + \cot(x)\arctan(x)\right)$$

Example 3

(I) Let $f(x) = \log_a(3x^2 - 2)$. For what value of a is $f'(1) = 3$?

$$f'(x) = \frac{6x}{\ln(a)(3x^2-2)} \qquad f'(1) = \frac{6}{\ln(a)}$$

- If $f'(1) = 3$ then $\ln(a) = 2$, so $a = e^2$.

(II) On what interval(s) is the function $f(x) = \dfrac{\ln(x)}{x}$ increasing, decreasing, concave upward and concave downward?

$$\frac{dy}{dx} = \frac{1-\ln(x)}{x^2} \qquad \frac{d^2y}{dx^2} = \frac{-3+2\ln(x)}{x^3}$$

- Increasing on $(0, e)$, decreasing on (e, ∞).
- Concave up on $(e^{1.5}, \infty)$, concave down on $(0, e^{1.5})$.

Example 4, Logarithmic Implicit Differentiation!

Consider the curve defined by the equation $y\sqrt{x^2+3} = x^y$.
Find the equation of the tangent line at the point $(1, \frac{1}{2})$.

$$\ln\left(y(x^2+3)^{1/2}\right) = \ln\left(x^y\right) \qquad \ln(y) + \frac{1}{2}\ln(x^2+3) = y\ln(x)$$

$$\frac{y'}{y} + \frac{2x}{2(x^2+3)} = \frac{y}{x} + y'\ln(x) \qquad \frac{y'}{y} - y'\ln(x) = \frac{y}{x} - \frac{x}{x^2+3}$$

$$y' = \frac{\frac{y}{x} - \frac{x}{x^2+3}}{\frac{1}{y} - \ln(x)} \qquad y'\Big|_{(x,y)=(1,\frac{1}{2})} = \frac{\frac{1}{2}-\frac{1}{4}}{2-0} = \frac{1}{8}$$

Answer: $\boxed{y - \frac{1}{2} = \frac{1}{8}(x-1)}$ or $\boxed{y = \frac{1}{8}x + \frac{3}{8}.}$